An Investigation of the Time Required for Control of Structures

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The minimum time required for accomplishing a rigid-body maneuver of a flexible structure by means of a finite number of unbounded inputs is investigated. The control task is accomplished by driving a finite number of structure modes from an initial state of zero to a desired final state in a given time interval. A minimum-effort control strategy is used so that uncontrolled higher modes will not be excited excessively by the control inputs. It is found that for less than a certain time interval for control, it is not possible to decrease the amount of spillover energy in uncontrolled modes at the end of the control interval by driving more flexible modes to zero. This time interval is identified as the minimum time required for control of flexible structures, and it is related to the time required for waves to travel through the structure. For one-dimensional second-order systems, the minimum time is equal to the time required for waves to travel between adjacent pairs of actuators. A similar result is found for fourth-order systems. Because the period of the lowest flexible mode is equal to the time required for a wave to make a round trip throughout the structure, the minimum time for control is in general some fraction of the period of the lowest flexible mode, which is determined by the number and placement of actuators.

Introduction

ACTIVE vibration control of flexible structures is motivated by a need to drive them to a quiescent final state more quickly. The time required to complete a control task will have a significant effect on the performance of flexible spacecraft in many cases, so that minimum-time control is of interest. Control theory for linear systems has been applied to the control of structures, often with a truncated set of modal displacements and velocities as states. One result from control theory is that the time required to drive a controllable finite-dimensional linear system from an initial state to a desired final state depends only on the bounds on the inputs, in the absence of state constraints. If the inputs are unbounded, the time interval for control can be made arbitrarily short.[1] It is tempting to assume that this result holds for flexible structures controlled by means of a finite number of inputs. If this is true, then it is attractive to use as few actuators as possible for control, as this reduces hardware requirements without degrading performance.

However, it is evident from wave-propagation concepts that structures, which are infinite-dimensional systems, cannot be controlled in an arbitrarily short time interval by a finite number of inputs applied at discrete points. Indeed, the motion of a point on a structure cannot be influenced by an actuator in less time than is required for a wave originating at the actuator to travel to the point, no matter what the actuator force level is. For this reason, the time required for waves originating at the actuators to reach all points on the structure must be a lower bound on the time required for control, even when inputs are unbounded. This is true in spite of the fact that any finite number of modes of the structure, with a given set of inputs, may constitute a controllable finite-dimensional system.

The distinction between the control model, consisting of a finite number of modes, and the actual structure, which possesses an infinity of modes, is of considerable importance in minimum-time control applications. If the time interval allowed for control of a structure is shorter than the time required for wave propagation from actuators to all points on the structure, the controlled modes may well reach their desired final states. However, there must be portions of the structure that have not yet been influenced by the actuators. This fact will be evident if the response of the structure is determined with enough higher uncontrolled modes included in the response calculations. The fact that higher modes must be excited to a significant extent to account for the discrepancy between the response of the control model and the actual structure response indicates that control spillover must be considered in minimum-time control applications. The question to be addressed is not how quickly a given finite-dimensional model of a structure can be driven from one state to another by a given set of inputs, but instead how quickly a model of the structure which includes enough modes to accurately represent the response of the controlled structure can be driven from a given initial state to the desired final state.

This paper is concerned with determining the time required to carry out a rigid-body translation of a flexible structure using a finite number of discrete unbounded inputs. This is done for two simple structures: a second-order structure such as a rod in axial vibration, a shaft in torsional vibration, or a string in transverse vibration; and a fourth-order system such as a slender beam in flexural vibration.

Determining the Time Required for Control

Ordinarily in the control of structures, the amount of energy in higher uncontrolled modes due to control spillover can be decreased by increasing the number of controlled modes. This is true if the time interval for the control task is reasonably large. However, if the time interval for control is shorter than the time required for waves from the actuators to travel throughout the structure, this will not be true. The implication is that the infinity of modes cannot be driven to an arbitrary set of final states, even though an arbitrarily large finite number of modes may constitute a controllable system. Hence, it will not be possible to make the spillover energy in the structure at the end of the control task arbitrarily small by increasing the number of modes in which the energy is driven to zero. It will be shown in this paper that increasing the number of modes that are driven to zero in a given time interval can cause the spillover energy in uncontrolled modes at the end of the control task to increase rapidly.

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These considerations suggest an approach for investigating the amount of time required for carrying out a rigid-body translation or rotation of a flexible structure by means of a finite number of discrete actuators. For a given time interval, a control history can be obtained which accomplishes the rigid-body maneuver and drives the energy in a certain number of flexible modes to zero. Then, with this control history established, the response of higher modes can be obtained so that the amount of spillover energy at the end of the control history can be determined. Next, the number of flexible modes in which the energy is to be driven to zero can be increased, without changing the time interval for control. Based on the control history that is obtained to accomplish this, the spillover energy at the end of the control task can again be determined. If the time interval is larger than the minimum time required to accomplish the desired control task, increasing the number of controlled modes should decrease the amount of spillover energy at the end of the control history. However, if the time interval is shorter than the minimum amount of time required, it will not be possible to drive the spillover energy to zero by increasing the number of controlled modes.

It is already evident that the time required for accomplishing a given control task of this type on a structure must be at least as great as the time required for waves to travel from the actuators throughout the structure. However, it remains for us to determine exactly what the time required for a given control task is. The answer to this question is not obvious, especially for dispersive systems in which the wave speed is dependent on the wavelength. This question is addressed in the following sections.

Second-Order Systems

The motion of a class of one-dimensional flexible systems is governed by a partial differential equation in the form[2]

$$-\frac{\partial}{\partial x}\left[s(x)\frac{\partial u(x,t)}{\partial x}\right]+m(x)\frac{\partial^2 u(x,t)}{\partial t^2}=f(x,t) \tag{1}$$

Rods in axial vibration, strings in transverse vibration, and shafts in torsional vibration belong to this class of systems. The displacement $u(x,t)$ is axial, transverse, or angular displacement as a function of axial position x and time t. The stiffness term $s(x)$ is an axial or torsional stiffness, or tension in the case of strings. The mass term $m(x)$ is the mass or the polar mass moment of inertia per unit length, and the force $f(x,t)$ is a distributed axial or transverse force or axial torque per unit length.

If the system is unrestrained at the ends, the displacement $u(x,t)$ must satisfy the free-end boundary conditions

$$s\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0}=s\frac{\partial u(x,t)}{\partial x}\bigg|_{x=L}=0 \tag{2}$$

If the system is uniform, both s and m are constant. By the expansion theorem, the displacement $u(x,t)$ can be represented in terms of the modes $\phi_r(x)$ and the modal displacements $q_r(t)$ by the sum

$$u(x,t)=\sum_{r=0}^{\infty}\phi_r(x)q_r(t) \tag{3}$$

in which the rigid-body mode $\phi_0(x)$ is given by

$$\phi_0(x)=\frac{1}{\sqrt{mL}} \tag{4}$$

and the flexible modes are

$$\phi_r(x)=\sqrt{\frac{2}{mL}}\cos\frac{r\pi x}{L}, \qquad r=1,2,\ldots \tag{5}$$

The natural frequencies of the unrestrained system are

$$\omega_r=r\pi\sqrt{\frac{s}{mL^2}}, \qquad r=0,1,\ldots \tag{6}$$

The modes have been normalized so that the orthogonality relations

$$\int_0^L m\phi_r(x)\phi_s(x)\,\mathrm{d}x=\delta_{rs} \tag{7a}$$

$$\int_0^L s\frac{\partial\phi_r(x)}{\partial x}\frac{\partial\phi_s(x)}{\partial x}\,\mathrm{d}x=\lambda_r\delta_{rs} \tag{7b}$$

are satisfied, where δ_{rs} is the Kronecker delta and $\lambda_r=\omega_r^2$ is the rth eigenvalue.

It should also be noted that the second-order systems described here also admit a solution in terms of waves in the form

$$u(x,t)=U_1(x-ct)+U_2(x+ct) \tag{8}$$

where $c=\sqrt{s/m}$ is the wave speed at which the wave profiles U_1 and U_2 are traveling to the right and to the left, respectively.[2]

In this study, a system controlled by two discrete force actuators located at the ends is considered. This gives a distributed force profile of the form

$$f(x,t)=F_1(t)\delta(x)+F_2(t)\delta(x-L) \tag{9}$$

where $\delta(x)$ is a Dirac delta function. Using the expansion theorem, Eq. (3), in the equation of motion, Eq. (1), multiplying by $\phi_s(x)$, and making use of the orthogonality relations, Eqs. (7), results in the modal equations of motion

$$\ddot{q}_0(t)=\frac{1}{\sqrt{mL}}[F_1(t)+F_2(t)] \tag{10}$$

and

$$\ddot{q}_r(t)+\lambda_r q_r(t)=\sqrt{\frac{2}{mL}}[F_1(t)+(-1)^r F_2(t)], \qquad r=1,2,\ldots \tag{11}$$

These can be arranged in state-space form, with modal displacements and velocities as states. With the state vector

$$\boldsymbol{x}=[q_0(t)q_1(t)\ldots\dot{q}_0(t)\dot{q}_1(t)\ldots]^T \tag{12}$$

the state equations take the familiar form

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u} \tag{13}$$

with

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ -\boldsymbol{\Lambda} & \boldsymbol{0}\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{b}_1 & \boldsymbol{b}_2\end{bmatrix},\quad \boldsymbol{u}=\begin{Bmatrix}F_1(t)\\ F_2(t)\end{Bmatrix} \tag{14}$$

in which $\boldsymbol{I}$ is the unit matrix, $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_0,\lambda_1,\ldots)$ is a diagonal matrix of eigenvalues, and

$$\boldsymbol{b}_1=\sqrt{\frac{2}{mL}}[\sqrt{1/2}\quad 1\quad 1\quad 1\quad 1\ldots]^T \tag{15a}$$

$$\boldsymbol{b}_2=\sqrt{\frac{2}{mL}}[\sqrt{1/2}\quad -1\quad 1\; -1\quad 1\ldots]^T \tag{15b}$$

Carrying out a rigid-body translation of the system, with the system initially at rest, amounts to driving the states from the initial conditions $\boldsymbol{x}(t_i)=[0,0,\ldots]^T$ to the final values of $\boldsymbol{x}(t_f)=[q_0,0,0,\ldots]^T$, where the value of q_0 is chosen to yield a rigid-body translation of the desired magnitude. It can be shown that this is a controllable system when any finite number n_c of modes is included in the control model. Hence, any finite number of controlled modes can be driven from the zero initial state to the desired final state in an arbitrary time interval, as stated in the Introduction. The resulting final states of the uncontrolled modes remain to be determined. A natural measure of the uncontrolled modes' deviation from their desired final states of zero at the end of the control history is the spillover energy in the uncontrolled modes, given by

$$E_{sp} = \frac{1}{2} \sum_{r=n_c+1}^{\infty} [\dot{q}_r^2 + \lambda_r q_r^2] \tag{16}$$

The two terms in E_{sp} are obtained by using the expansion theorem and the orthogonality relations in the expressions for kinetic and potential energy

$$T = \frac{1}{2} \int_0^L m\dot{u}^2(x,t)\, \mathrm{d}x \tag{17a}$$

$$V = \frac{1}{2} \int_0^L s \left[\frac{\partial u(x,t)}{\partial x}\right]^2 \mathrm{d}x \tag{17b}$$

The spillover energy E_{sp} will be monitored as the number of controlled modes and the time interval for control, $\Delta t = t_f - t_i$, are varied. The spillover energy depends on the control history, which is not yet determined when only the initial and final states of controlled modes are specified. To avoid excessive excitation of uncontrolled modes, without explicitly taking them into account, the control history that minimizes the actuator effort as measured by the performance functional

$$J = \int_{t_i}^{t_f} \boldsymbol{u}T(t)\boldsymbol{u}(t)\, \mathrm{d}t \tag{18}$$

is chosen. This constitutes an optimal control problem whose solution is[3]

$$\boldsymbol{u}(t) = -\boldsymbol{B}^T\boldsymbol{p}(t) \tag{19}$$

where $\boldsymbol{p}(t)$ is a vector of co-states which, with the states $\boldsymbol{x}_c$ corresponding to controlled modes, are governed by the equation

$$\begin{Bmatrix} \dot{\boldsymbol{x}}_c \\ \dot{\boldsymbol{p}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A}_c & -\boldsymbol{B}_c\boldsymbol{B}_c^T \\ \boldsymbol{0} & -\boldsymbol{A}_c^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_c \\ \boldsymbol{p} \end{Bmatrix} = \tilde{\boldsymbol{A}}_c \begin{Bmatrix} \boldsymbol{x}_c \\ \boldsymbol{p} \end{Bmatrix} \tag{20}$$

where $\boldsymbol{A}_c$ and $\boldsymbol{B}_c$ only include the entries in $\boldsymbol{A}$ and $\boldsymbol{B}$ corresponding to the controlled modes. The solution to this equation is obtained in terms of a matrix exponential that can be partitioned as follows:

$$\begin{Bmatrix} \boldsymbol{x}_c(t_f) \\ \boldsymbol{p}(t_f) \end{Bmatrix} = e^{\tilde{\boldsymbol{A}}_c(t_f - t_i)} \begin{Bmatrix} \boldsymbol{x}_c(t_i) \\ \boldsymbol{p}(t_i) \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_c(t_i) \\ \boldsymbol{p}(t_i) \end{Bmatrix} \tag{21}$$

If $\boldsymbol{x}_c(t_i)$ and $\boldsymbol{x}_c(t_f)$ are specified as described earlier for a rigid-body translation of the system, $\boldsymbol{p}(t_i)$ can be obtained from the upper partition of the preceding equation as

$$\boldsymbol{p}(t_i) = \phi_{12}^{-1}[\boldsymbol{x}_c(t_f) - \phi_{11}\boldsymbol{x}_c(t_i)] = \phi_{12}^{-1}\boldsymbol{x}_c(t_f) \tag{22}$$

This result is needed for determining the response of the uncontrolled modes to the control history. From Eqs. (13) and (19), we see that these obey an equation similar to Eq. (20):

$$\begin{Bmatrix} \dot{\boldsymbol{x}}_{uc} \\ \dot{\boldsymbol{p}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A}_{uc} & -\boldsymbol{B}_{uc}\boldsymbol{B}_c^T \\ \boldsymbol{0} & -\boldsymbol{A}_c^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_{uc} \\ \boldsymbol{p} \end{Bmatrix} = \tilde{\boldsymbol{A}}_{uc} \begin{Bmatrix} \boldsymbol{x}_{uc} \\ \boldsymbol{p} \end{Bmatrix} \tag{23}$$

where $\boldsymbol{x}_{uc}$, $\boldsymbol{A}_{uc}$, and $\boldsymbol{B}_{uc}$ contain entries related to uncontrolled modes. Final states for uncontrolled modes are obtained from

$$\begin{Bmatrix} \boldsymbol{x}_{uc}(t_f) \\ \boldsymbol{p}(t_f) \end{Bmatrix} = e^{\tilde{\boldsymbol{A}}_{uc}(t_f - t_i)} \begin{Bmatrix} \boldsymbol{x}_{uc}(t_i) \\ \boldsymbol{p}(t_i) \end{Bmatrix} \tag{24}$$

These final states are inserted into Eq. (16) to obtain the spillover energy at the end of the control history.

At this point, a procedure has been presented for determining the spillover energy that results from carrying out a rigid-body translation of a uniform second-order flexible structure by means of two actuators located at the ends. The actuators drive any given finite number n_c of flexible modes to zero at the end of the control history, with a minimum of control effort as measured by the performance index of Eq. (18). Because of the symmetry of both the system and the control objective, the antisymmetric modes are not excited by the control history that results from this procedure, and need not be retained in the calculations. This can be shown by defining symmetric and antisymmetric control inputs in terms of $F_1(t)$ and $F_2(t)$, and rewriting the state equations in terms of these. Then it can be seen that the symmetric and antisymmetric problems, in which each consists of a set of modal states and one control input, are uncoupled. The matrix exponentials required in this investigation can be obtained in closed form to avoid numerical difficulties. Finally, since any finite number of uncontrolled modes can be included in Eq. (24), the spillover energy can be calculated for only a few uncontrolled modes at a time, and accumulated to obtain the total spillover energy for a given control history. When the amount of spillover energy contributed by these groups of modes falls to a level that can be considered negligible, the total spillover energy has been determined to a sufficient level of accuracy.

In this investigation, this typically required evaluating the spillover energy in about 10 symmetric modes before convergence occurred, when the time allowed for control was reasonably large. As the control time was decreased, the number of modes that had to be considered increased, to about 30 symmetric modes for the shortest times considered here. Considering the antisymmetric modes, this implies that the response was essentially contained within the lower 60 modes of the system.

Figure 1 is a plot of the logarithm to the base 10 of the spillover energy vs the time interval for control, where the time interval Δt, is normalized by dividing it by the period of the lowest flexible mode, T_1. Each curve is for a different number of controlled modes. It can be seen that, for large Δt, the curve obtained by controlling only the rigid-body mode is above the rest, indicating that spillover energy can be significantly decreased by controlling more modes, as expected. However, as the time interval decreases and approaches a value of one-half the period of the lowest flexible mode, a very

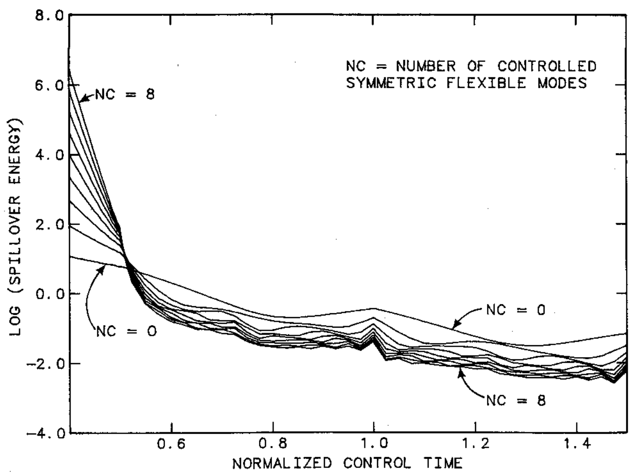

Fig. 1 Spillover energy vs normalized control time for a second-order system.

interesting thing happens. All of the curves cross one another, so that, *for Δt less than one-half the period of the first flexible mode, controlling more modes with a minimum of control effort results in an increase in the amount of spillover energy*. The implication of this is that, for time intervals shorter than this, it is not possible to decrease the degree to which higher modes are excited by including more modes in the control model. From Fig. 1, it is clear that the spillover energy increases rapidly as more modes are controlled, a fact that becomes more striking when it is recognized that the spillover energy is plotted on a logarithmic scale. Hence, it can be said that one-half the period of the lowest mode is the shortest time interval in which a rigid-body translation of this flexible system can be carried out by two actuators located at the ends.

Figure 2 shows the region of the graph of Fig. 1 in the neighborhood of the minimum time for control. Here it is evident that all of the curves do not pass through a single point, but their crossings approach the time interval of one-half of the period of the lowest flexible mode as the number of controlled modes increases.

Figure 3 contains plots of the control histories of the actuators for the time interval $\Delta t = t_{\min} = \sqrt{mL^2/s}$, with increasing numbers of controlled modes. It is evident that, as the number of controlled modes is increased, the control history converges in a manner reminiscent of Fourier series to the minimum-time control history

$$F_1(t) = F_2(t) = \frac{q_0}{2}\sqrt{\frac{s}{L}}[\delta(t) - \delta(t - t_{\min})] \qquad (25)$$

where $\delta(t)$ is a Dirac delta function. Here the distance translated is equal to $q_0\sqrt{1/mL}$, as seen from Eqs. (3) and (4), and the amplitude of the Dirac delta functions can be determined by considering the impulse that must be imparted to the system to cause it to translate the given distance in the time $t_{\min}$.

The effect of the first Dirac delta functions in the control history is to instantaneously move the ends of the system one-half of the total distance translated. As a result of this, two square wavefronts are propagated across the system from each end at the wave speed of the system, which is equal to $c = \sqrt{s/m}$. When they meet in the center, they superimpose on each other, so that the center instantaneously moves the entire distance to be translated by the system, so that it is in its desired final position. The two wavefronts, having "passed through" one another continue across the system until they reach the ends, at which point the second Dirac delta functions in the control history stop the translation of the system at the desired final position.

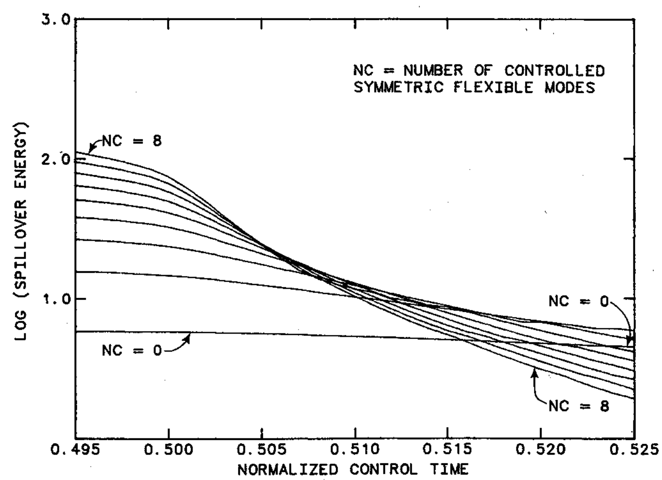

Fig. 2 Enlarged view of Fig. 1 in the neighborhood of $\Delta t = t_{\min} = 0.5T_1$.

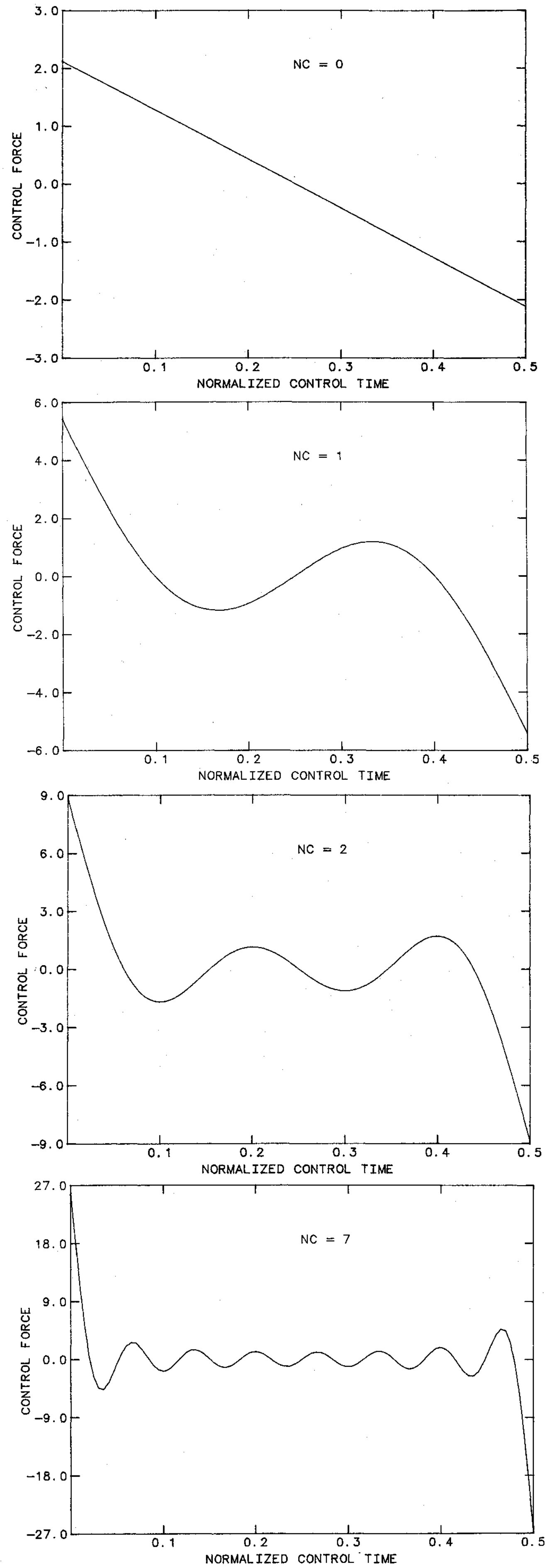

Fig. 3 Control histories with 0, 1, 2, and 7 flexible modes controlled.

From another point of view, the effect of the first Dirac delta functions in the control history is to impart to the system the momentum necessary for the desired translation. The propagation of waves throughout the system carries the effect of this momentum throughout the entire system, and the second Dirac delta functions remove the momentum from the system, leaving it at the desired final location.

It must be pointed out that this control history can only be regarded as having theoretical significance, as it could not be implemented on a real system due to actuator force level constraints. Also, the deformation that would theoretically result could not be reproduced by a real material, as it is characterized by discontinuities in the displacement at the wavefronts. However, the effect of actuator and material limitations is to increase the time interval required for control above the theoretical minimum obtained here. Minimum-time control in the presence of actuator and strength constraints will be dealt with in a future paper.[4]

This experiment for determining the time required for control can be repeated on a system with two actuators at the ends and an additional one in the center. When this is done, it is found that the time required is divided in half, and the control history is essentially the same as in Eq. (25). In this case, the time required for control is equal to the time required for waves to travel between actuators, as before. In fact, this result can be generalized for an arbitrary number of actuators in arbitrary locations, so that the time required for control is equal to the time required for waves to travel between the most widely separated pair of adjacent actuators. Hence, the time required for control is reduced by using a greater number of actuators, as it is simply proportional to the greatest distance between actuators. To relate this result to the discussion in the Introduction, it has been determined that the time required for control in this case is equal to twice the lower bound mentioned there, as waves must not only be able to reach all points in the structure in the minimum time interval, but they must also be able to travel the entire distance between adjacent pairs of actuators.

It is evident that the relationship between the time required for control and the period of the lowest flexible mode is no coincidence, when the "wave train closure principle" is considered.[5] This principle states that the period of the lowest flexible mode of the system is equal to the time required for a wave to make one round trip through the system. With two actuators located at the ends of the system, the waves they produce must only travel one-half of a round trip, as they must only reach the opposite end. Hence, the time required for control is equal to one-half of the period of the lowest flexible mode.

Now that the time required for control of a uniform second-order system has been investigated, in the next section the time required for control of fourth-order systems is addressed.

Fourth-Order Systems

The partial differential equation governing the motion of a slender beam, when the effects of shear deformation and rotatory inertia are neglected, is[2]

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+m(x)\frac{\partial^2 w(x,t)}{\partial t^2}=f(x,t) \tag{26}$$

where $EI(x)$ is the flexural rigidity of the beam, $w(x,t)$ the transverse deflection, $m(x)$ the mass per unit length, and $f(x,t)$ the transverse force per unit length. The displacement of a uniform beam of length L that is free at both ends satisfies the boundary conditions

$$EI\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=-L/2}=EI\frac{\partial^3 w(x,t)}{\partial x^3}\bigg|_{x=-L/2}=0 \tag{27a}$$

$$EI\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=L/2}=EI\frac{\partial^3 w(x,t)}{\partial x^3}\bigg|_{x=L/2}=0 \tag{27b}$$

if the origin of the coordinate system is located at the center of the beam. The motion of the beam can be expressed in terms of the natural modes of vibration using the expansion theorem as in Eq. (3). If a rigid-body translation of this beam is intended, it can be recognized at this point that the interest is in the symmetric modes of the beam, as was the case with the uniform second-order system of the previous section. The symmetric rigid-body mode is identical to the one in Eq. (4), and the symmetric flexible modes are given in normalized form as

$$\phi_r(x)=\sqrt{\frac{2}{mL}}\left[\frac{\cosh\frac{\beta_r L}{2}\cos\beta_r x+\cos\frac{\beta_r L}{2}\cosh\beta_r x}{\left(\cosh^2\frac{\beta_r L}{2}+\cos^2\frac{\beta_r L}{2}\right)^{1/2}}\right], \quad r=1,2,\ldots \tag{28}$$

where the β_r are the positive roots of the transcendental equation[2]

$$\cos\beta_r L\ \cosh\beta_r L=1 \tag{29}$$

The nonzero natural frequencies ω_r are given by

$$\omega_r=\beta_r^2\sqrt{\frac{EI}{m}}, \qquad r=1,2,\ldots \tag{30}$$

The kinetic energy expression for the beam can be obtained from the kinetic energy in Eqs. (17) by substituting $\dot{w}(x,t)$ for $\dot{u}(x,t)$, and the potential energy is given by

$$V=\frac{1}{2}\int_0^L EI\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]^2 dx \tag{31}$$

It can be verified that beams also admit wave motion, by substituting the displacement

$$w(x,t)=\cos k(x-vt) \tag{32}$$

into Eq. (26), where k is the wave number and v the wave speed. Upon doing so, it is found that the wave speed must be $v=k\sqrt{EI/m}$, so that the wave speed is proportional to the wave number, or inversely proportional to the wavelength[2]. A general wave profile traveling on a beam can be decomposed into sinusoidal components by Fourier analysis, with each component traveling at a different wave speed. This causes the wave profile to change shape as it travels, so that the medium is termed dispersive. Note that the wave speed approaches infinity in direct proportion to the wave number, or as the wavelength approaches zero.

The rest of the development for the beam parallels the development in the previous section very closely, so that it is omitted here for brevity. Again, the initial objective is to carry out a rigid-body translation of the beam using only two actuators located at the ends of the beam. Using the same steps as before, the plot in Fig. 4 is obtained, which corresponds to Fig. 1 in the case of the second-order system. Again, different curves have been plotted for different numbers of controlled modes, with the time interval divided by the period of the lowest flexible mode on the horizontal axis and the logarithm of the amount of spillover energy on the vertical axis.

A qualitative similarity between Fig. 4 and Fig. 1 is immediately apparent, in that for large Δt, the spillover energy decreases as the number of controlled modes increases, and for small Δt the spillover energy increases rapidly as the number of controlled modes increases. However, the transition between the two types of behavior is not as localized as in the case of the second-order system, so that the minimum time required for the control task is not so easily identified. Also, on looking at the control histories in the transition

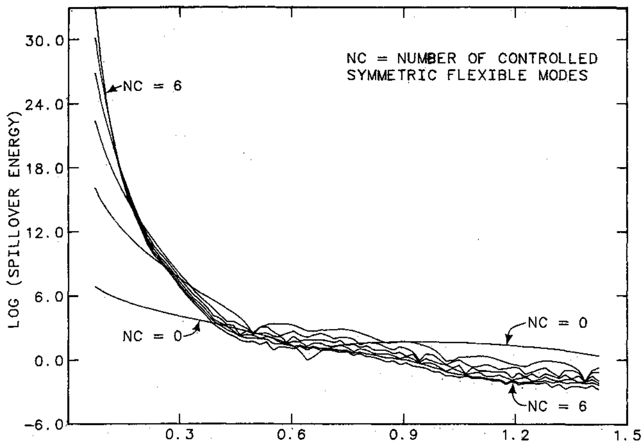

Fig. 4 Spillover energy vs normalized control time for a fourth-order system.

region, it is not obvious that what they are converging to is as easily recognizable as the control histories for the second-order system, as the number of controlled modes increases. Unfortunately, the dispersive nature of the beam complicates the problem so that the results are not interpreted as easily. Because the wave speed is proportional to the wave number, it might be expected that it would be possible to carry out a transverse translation of the beam in an arbitrarily short period of time by propagating waves of the necessary wave number across the beam. However, the energy per unit length associated with the motion of Eq. (32) can be shown to be proportional to the fourth power of the wave number, or the wave speed. This indicates that severe flexing of the beam results, so that material strength limits are likely to be quickly exceeded as the wave number increases. Hence, the idea that the wave speed can be made arbitrarily fast to accomplish the control task more quickly has questionable merit. Figure 4 indicates that, for small Δt, the spillover energy increases by a number of orders of magnitude as the time interval decreases, particularly with a larger number of controlled modes.

Another point that must be considered is that rotatory inertia and shear deformation are neglected in Eq. (26). If only rotatory inertia were considered, it is seen that the wave speed for a beam with any finite rotatory inertia cannot exceed the wave speed associated with longitudinal vibration, which is equal to the square root of the ratio of the elastic modulus to the mass density.[2] For this reason, it may be expected that the time required for longitudinal translation of a flexible beam, using longitudinally oriented actuators in the same locations, would be a lower bound on the time required for transverse translation. For a given beam having a specific cross-sectional rotatory inertia, this could be investigated using the same approach that was used here.

To determine how the time required for control depends on the actuator spacing, the number of actuators was increased and the study that resulted in Fig. 4 for two actuators was repeated. In each case, the actuators were evenly spaced along the beam, with two located at the ends. The results obtained were similar to those presented in Fig. 4, with the main difference being that smaller time intervals were involved. For comparison between the different cases, the time interval at which the curve for no controlled flexible modes crossed the curve generated with six symmetric flexible modes controlled, denoted by $\hat{t}_{min}$, was arbitrarily chosen as an estimate of the minimum time required for control. This is the smallest time interval for which controlling a large number of modes (in this case, the most allowed by computational constraints) results in as little spillover energy as controlling the rigid-body mode alone, so that this time interval approximates the minimum time required. From Fig. 2, it can be seen that this method of estimating the time required for control would result in an error of about 3.5% for the second-order case.

Table 1 Dependence of $\hat{t}_{min}$ on actuator spacing

Number of actuators	Actuator spacing, d/L	Minimum time estimate, $\hat{t}_{min}/T_1$	Minimum time approximation, $0.4(d/L)^{3/2}$
2	1.00	0.388	0.400
3	0.50	0.089	0.141
4	0.33	0.078	0.077
5	0.25	0.049	0.050
6	0.20	0.037	0.036

Table 1 contains the results of this investigation. Although this procedure presents a limited amount of approximate information, it appears from these results that, as the number of actuators increases, the time required for control is very nearly proportional to the distance between actuators raised to the power 1.5. The final column contains values obtained from a simple formula which is seen to approximate $\hat{t}_{min}/T_1$ very well in all cases except the one in which only three actuators are used. The most noteworthy result of this brief investigation is that the time required for control of a system having flexural rigidity appears to decrease more quickly as the distance between actuators decreases than in the second-order case, where the time required for control is simply proportional to the distance between actuators. This result is in agreement with what intuition might suggest for a system having flexural rigidity.

Conclusions

In this paper, the amount of time required for carrying out the rigid-body translation of flexible, one-dimensional, second- and fourth-order systems using a finite number of unbounded discrete inputs was investigated. This was done by determining the control history that accomplishes a rigid-body translation and drives the energy in a finite number of flexible modes to zero in a specified time interval, using a minimum of control effort. The spillover energy at the end of the control task was then calculated. It was seen for uniform second-order systems that, for a time interval greater than the time required for waves to travel between adjacent pairs of actuators, increasing the number of modes in which the energy was driven to zero decreased the spillover energy at the end of the maneuver, as would be expected. However, for time intervals shorter than this, increasing the number of modes in which energy was driven to zero caused the amount of spillover energy to increase rapidly. This indicates that the time required for waves to travel between actuators is the shortest possible time interval in which the control task can be accomplished. If two actuators are located at the ends of the structure, the minimum time required for control is equal to one-half of the period of the lowest flexible mode. As more actuators are added to the system, the time required for control decreases. The control history that accomplishes rigid-body translation of uniform second-order systems exactly, in the minimum possible time, has been obtained here for the two actuator case, and it consists of Dirac delta functions in time, which propagate square wavefronts through the system.

Slender beams in bending are fourth-order systems that have a wave speed that is inversely proportional to the wavelength of the wave being propagated, in contrast to second-order systems that have a constant wave speed. On investigating the spillover energy resulting from the minimum-effort control of beams in a specified time interval, results similar to those of the second-order case were found. However, presumably because of the variable wave speed, the transition between the case in which increasing the number of controlled modes decreases the spillover energy, and the case in which doing so increases the amount of spillover energy, occurs over a much wider range in the time interval for control, so that the identification of the minimum possible time for control is more difficult. It is evident, though, that the time interval for control cannot be made arbitrarily short, because this calls for extremely short wavelengths that involve unrealistically high amounts of strain energy in the beam. Also, when rotatory inertia effects are considered, it is seen that flexural waves cannot travel faster than longitudinal waves, so that the time required for transverse translation of a beam cannot be less than the time required for axial translation with actuators in the same locations. The time required for transverse translation of a uniform slender beam with two actuators located at the ends was estimated to be about 39% of the period of the first flexible mode. As additional actuators were added, the time required for control was seen to be approximately proportional to the distance between actuators raised to the power 1.5.

In this paper, it is found that in minimum-time control applications, the infinite-dimensional nature of structures cannot be ignored. For control of structures in a relatively long period of time, finite-dimensional models are adequate, as higher modes are not likely to be excited to a significant extent. However, if too short a time interval is allowed for control, no finite-dimensional model will be adequate. The inclusion of additional modes in the control model in an attempt to reduce the spillover energy can only result in higher modes being excited to an even greater extent, and these higher modes always exist for structures. This points out that the modal control approach has certain limitations in the minimum-time control of structures. The modal approach gives no indication that the control cannot be carried out arbitrarily quickly, unless the response of the uncontrolled modes is investigated adequately. The lower bound on the time required for control can be seen as a performance constraint that is due to the infinite-dimensionality or flexibility of the structure, rather than from more familiar sources, such as bounds on control inputs or state constraints.

The time required for control of a structure is closely related to the time required for wave travel throughout the structure. The "wave train closure principle" relates the wave travel time to the periods of the natural modes of vibration, by stating that a natural period of vibration is equal to the time required for a wave to make a round trip throughout a structure and close on itself with zero phase change.[5] Hence, it can be expected in general that the minimum time for control will be some fraction of the period of the lowest mode of a structure, which is determined largely by the number of actuators. Historically, a major motivation for research in the active control of structures has been the need to be able to effectively handle an overlap between the control bandwidth and the structure bandwidth. The results of this investigation indicate that the essential nature of flexible structures limits the extent to which such an overlap can take place.

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